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Patterns of Misunderstanding: An Integrative Model for Science, Math, and Programming

D. N. Perkins and **Rebecca Simmons**
Harvard Graduate School of Education

This article examines unifying factors among diverse problems of understanding in several fields. Certain misunderstandings in science, mathematics, and computer programming display strong structural analogies with one another. Even within one of these domains, however, not all misunderstandings are structurally similar. To explain the commonality and variety, four levels of knowledge are posited: (a) content, (b) problem-solving, (c) epistemic, and (d) inquiry. Through analysis of several examples, it is argued that misunderstandings have causes at multiple levels, with highly domain-specific causes predominant at the "content" level and somewhat more general causes at the other levels. The authors note that education characteristically neglects all but the content level, describe successful interventions at all levels, and urge more attention in education to integration across the levels.

Recent research in the learning of physics, mathematics, and programming tells a tale of similarity within diversity. Despite significant differences among these domains, patterns of misunderstanding appear in novices—and sometimes even in experts—that seem in many ways to reflect analogous underlying difficulties. In physics, for example, students typically solve problems by rote equation cranking. They first engage in a formulaic matching of the variables presented in the problem to equations and then perform standard algebraic transformations on the equations to solve for the unknowns (Chi, Feltovich, & Glaser, 1981; Chi, Glaser, & Rees, 1982; Larkin, McDermott, Simon, & Simon, 1980; White & Horwitz, 1987). Missing is the sense of the "deep structure" of the problem organized around key interpretive concepts such as conservation of energy.

In mathematics problem solving, one sees the same pattern of attention to surface similarities. Schoenfeld (1985) notes examples of students who characteristically perform meaningless calculations on a problem, with no attention paid to whether or not the particular approach is justified, or progress being made. Instead, students invoke schemata apparently based on such phenomena as recency or familiarity. Resnick discusses the inclination of students engaged in mathematics to attend to formal notations and rules for manipulating them without relating the rules to the semantics of the notations. This tendency to separate syntactic manipulation and

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underlying semantics results in the formulation of incorrect rules or "malrules" (Resnick, 1987). In computer programming as well, "template-bound" coding with no apparent accompanying attention to the underlying mechanisms of the primitives is a common and persistent stumbling block (Perkins & Martin, 1986; Perkins, Martin, & Farady, 1986).

Another anomaly frequently seen across domains is the use of notational expressions that somehow violate the semantics. This occurs in math when students make errors of distributivity; for example, the square root of $A^2 + B^2$ is said to equal A plus B (cf. Matz, 1982). Note that algebra refers to numbers—numbers provide its semantics, so to speak—and an effort to check the distributivity relation with numbers would quickly disclose its error. In programming, the same phenomenon occurs; one sees expressions such as "PRINT 5*" (in BASIC) as an attempt to output a row of five stars on the screen (Perkins & Martin, 1986). Students often do not check such expressions against the semantics of the programming language by hand-executing the expression. In the sciences, it is not uncommon for students to derive and report physically implausible or meaningless results—negative speeds or masses, for instance—because that is what the equation cranking has delivered.

In all these instances, it is as though the students have forgotten what the statements mean in terms of interpretive models—numbers in the case of algebra, machine actions in the case of programming, physical quantities with certain constraints in the case of physics—and instead write and manipulate expressions in ways that suit their hopes and intentions, unfettered by a constraining semantics.

A further problem exhibited by students can be characterized as an anthropomorhphism or animism regarding major elements in the domain. Naive animistic concepts are commonplace in physics. For instance, sources of forces need to be agent-like things that can "push back," such as springs or springy substances; rigid substances are seen as having no way to push back (Clement, 1987a,b; diSessa, 1983). Students' beliefs here are mistaken in two ways at once: First, one can argue that all objects are more or less springy. Second, in any case, the existence of an equal and opposite reaction force does not require springiness.

Likewise, in programming, many errors can be attributed to a conceptual "superbug" in which the computer "understands" the meaning and the intentions of the statements it processes (Pea, 1986). Thus, for example, given a variable name of LARGEST in a Pascal program, the computer will "know" to store the largest of a series of numbers it reads into that variable because it comprehends the semantic unit "largest" (Sleeman, Putnam, Baxter, & Kuspa, 1986).

In sum, among a surfeit of student misunderstandings, there appear to be some strong similarities across the domains of physics, math, programming, and no doubt others as well. This observation raises a number of intriguing questions: Can one characterize broadly parallel causes of these parallel misunderstandings? Moreover, in some degree, might the causes be not just parallel but identical—crosscutting kinds of knowledge that, because faulty in certain ways, undermine understanding in several domains at once? Such a stand would, of course, run counter to one current viewpoint that holds that expert performance is, by and large, domain specific. Finally, what educational implications would follow from identifying parallel and common causes?

In this article, we present a tentative attempt at an integrative model of misunderstandings in science, math, and programming, directed at answering these

questions. First of all, we identify four levels of knowledge implicated in misunderstandings. In the first of these, which is mostly domain specific, misunderstandings arise because of parallel causes in different domains. In the other three, which are to some extent crosscutting, misunderstandings arise in different domains because of the very same causes. Second, we identify a few typical patterns or syndromes of understanding—"gestalts" that arise frequently, involving several of the levels. We suggest that misunderstandings have causes at multiple levels that interlock to form these distinctive gestalts. Finally, we argue in light of this framework that education characteristically neglects large parts of certain levels of knowledge and virtually all of certain others. Better education calls for more thorough attention to each and every level.

Four Frames for Understanding

We find it heuristic to characterize deep understanding of a domain as involving four interlocked levels of knowledge: the content frame, the problem-solving frame, the epistemic frame, and the inquiry frame. The term "frame" simply serves as a reminder that each of these is a system of schemata internally coherent and partially independent of the other frames.

In brief, the four frames are the following (they will be discussed at length later):

Content frame. This contains the facts, definitions, and algorithms associated with the "content" of a subject matter, along with content-oriented metacognitive knowledge, such as strategies for monitoring the execution of an algorithm, or for memorization and recall. Associated with the frame are characteristic performances, including recall of facts and correct description of instances using the vocabulary of the domain in question.

Problem-solving frame. This contains domain specific and general problem solving strategies, beliefs about problem solving, and autoregulative processes to keep oneself organized during problem solving. Here again there are characteristic performances: the solving of conventional textbook problems, and sometimes "qualitative" problems as well.

Epistemic frame. This frame incorporates domain specific and general norms and strategies concerning the validation of claims in the domain. Within a well-developed domain, the "facts" in the content frame are valid by the measure of the norms in the epistemic frame. Again there are characteristic performances: giving evidence, explaining rationales, proposing tests of claims.

Inquiry frame. This frame includes domain specific and general beliefs and strategies that work to extend and to challenge the knowledge within a particular domain. The characteristic performance is critical and creative thinking that questions the boundaries of the domain.

Note that each frame contains a variety of kinds of knowledge. Moreover, each frame encompasses knowledge ranging from the very domain-specific to the quite domain-general. Even within the content frame, as we define it, there are kinds of knowledge that could be described as metacognitive or strategic. For example, the content frame not only includes facts and algorithms, but strategies for memorization and recall. Also, within each frame there will be both declarative and "compiled" procedural knowledge (cf. Anderson, 1983). As will become apparent, our analysis is not concerned with the detailed cognitive mechanisms by which information in the four frames is activated and applied. Our framework requires

no mechanisms particular to it; we assume the sorts of activation and application processes that are commonly involved in all sorts of information processing. Rather, we focus on issues of repertoire, arguing that students need knowledge from the four frames to understand concepts deeply. Further, we assert that misunderstandings can in part be explained by a shallow repertoire in the noncontent frames and that appropriately designed education can do much to foster understanding by addressing all the frames and their interactions.

What are these frames and how do they relate to the mechanisms of cognition? Basically, the frames are simply categories that distinguish important kinds of knowledge. Many cognitive activities of students and professionals deploy knowledge from two or more frames in a complex mix. For example, a student solving a conventional textbook problem will draw heuristics and self-monitoring strategies from the problem-solving frame but with continuous accessing of information from within the content frame. A qualitative problem may press an able student to draw principles and strategies from the epistemic and inquiry frames, again with continuous reference to the content frame.

The four frames have been described in terms of what they should contain. But, of course, often these frames contain instead naive and reductive notions. Sometimes they may contain hardly any notions at all—a student may have no idea what the epistemic foundations of a particular subject matter are, or how to deploy the domain in a spirit of inquiry.

It is easy to see that ideal versions of the four frames would contribute to a very sophisticated understanding of a domain. One might question, however, whether all four are relevant to our aspirations for students' understanding. Do they not go too far? Could not a student perfectly well evade a number of specific misunderstandings regarding particular concepts, rules, and physical laws without nearly so sophisticated an armamentum? Moreover, much of the knowledge in the problem-solving, epistemic, and inquiry frames is quite general in nature; does this analysis not run counter to the current evidence that expertise is heavily rooted in domain-specific knowledge (e.g., Chase & Simon, 1973; Chi, Feltovich, & Glaser, 1981; Chi, Glaser, & Rees, 1982; Glaser, 1984; Newell & Simon, 1972; Schoenfeld & Herrmann, 1982)? For all these reasons, the four frames may appear to be an unlikely basis for a deeper look at the roots of students' misunderstandings.

Despite these concerns, however, we urge that the four frames play a fundamental role. We will argue that, contrary to appearances, evading typical misunderstandings of the content of a domain in any real sense *requires* the kinds of higher order sophistication represented by the noncontent frames. The evidence on expertise notwithstanding, we will argue that understanding intrinsically involves domain-general considerations such as those articulated in the noncontent frames.

A Radical Example

Before describing the four frames in more detail, we can begin on these arguments by discussing a single example to preview the general approach. As mentioned earlier, students commonly and inappropriately apply a principle of distributivity to the radical sign. For example, an algebra student may write: $\sqrt{(A^2 + B^2)} = A + B$. Now the key question is: Why has the student fallen into this misconception?

Thinking purely in terms of the content frame for elementary algebra, one might simply say that there is a gap in the student's knowledge base about algebra. The

student does not happen to know that distributivity does not apply to the radical. The student either believes the principle *does* apply, or, although uncertain, presumes that it does by explicit or tacit analogy with the distributivity of multiplication over division and of exponentiation over multiplication.

But can one really stop with the content frame? On the contrary, the circumstances suggest that general procedures attributable to a problem-solving frame play a role here. The uncertain student, who uses analogy to generate a rule, is deploying a general strategy, one that students use in many contexts. Without the analogical strategy, there would be no false rule to worry about; the student simply would not know what to do. Unfortunately, it seems that the student's problem-solving frame lacks a complementary strategy. A strong problem-solving frame for math, the hard sciences, and programming embodies a conservative strategy to filter out false analogies: All steps need grounding; when uncertain, check the soundness of an idea rather than just proceeding on speculation. A student lacking this conservative principle responds somehow, anyhow, in order to proceed with the problem, producing "repairs" that usually introduce more error (cf. Brown & VanLehn, 1980; VanLehn, 1981).

Now let us consider whether there is a role for the epistemic frame in this example. Suppose the student does feel uncertain about the distributivity rule yielded by analogy-making. What might the student's epistemic frame say that would encourage, or discourage, an effort to check the rule? Many students' epistemic frames seem to treat rules of a formal system like algebra rather like "rules of the game." Somewhere, someone made up, or figured out, the rules, which are basically to be learned, not checked. If one has to check, one asks the teacher or looks the rule up in a book. A very different and less reductive epistemology would recognize that any notational system, such as that of algebra, has a semantic basis that entails the rules. In the case of algebra, the semantic basis is real number arithmetic (ultimately, complex number arithmetic), and one can always check a proposed algebraic relation by testing it with numbers. A student with a strong sense that numbers provide the semantic foundation for algebra is considerably more likely to see checking with numbers as a reasonable and rewarding course of action.

As to the inquiry frame, it must be plain that proceeding on speculation discloses no critical posture at all toward the algebraic system and the (invalid) extension of it offered by the false distributivity rule. It is as though the student's tacit inquiry frame reads like this: Go by what you know; if you don't know, ask; if you can't ask, guess. In other words, "inquiry" takes the reductive form of interrogating official sources.

With this particular example reviewed, let us return to the first of the general questions raised earlier: Could not a student evade this misunderstanding without help from the noncontent frames? Well, of course a student *could.* The student might simply know by rote that the rule was false and hence not make the mistake. This kind of escape from error, however, does not in itself constitute understanding but simply skill with the rituals of algebra. To appreciate why there is a rule that might hold, but does not, the student must encompass more. The student must see how analogy motivates a certain rule; this understanding seems to implicate the problem-solving frame. The student must see how a check against real numbers invalidates the rule, and must appreciate that this is not merely a ritualized way of

checking expressions but rather just the right check to make—because the very rationale for the rule structure of algebra is inherited from the real numbers. This implicates the epistemic frame. Perhaps the student could understand all this without a strong inquiry frame, but at least the other two seem necessary.

Now consider the second question raised earlier: How does all this jibe with the research on expertise that argues that expertise is highly domain specific (e.g., Chi, Feltovich, & Glaser, 1981; Glaser, 1984; Newell & Simon, 1972; Schoenfeld & Herrmann, 1982). In reality, there is no conflict. First of all, the dependency of expertise on domain-specific knowledge of the content frame and those domain dependent aspects of the problem-solving frame is part of the present model. For instance, the student is in no position to check or even generate the false radical rule unless the student has the beginnings of an acquaintance with algebra and quite a good sense of number. Second, the dependency on domain-specific knowledge in no way entails that there are not other more crosscutting knowledge systems at work—the many general aspects of the problem-solving, epistemic, and inquiry frames.

Third, and perhaps most important, expertise and understanding are not the same thing, at least if we take expertise in a narrow sense. It is perfectly possible to be a facile handler of textbook algebra problems without really understanding algebra; one has merely become very good at the rituals of algebra. Likewise, it is perfectly possible to understand quite well the grounds of algebra, but to proceed with algebra problems in a halting and roundabout way because one has not yet acquired the "compiled" contextualized procedural knowledge characteristic of expertise (cf. Anderson, 1983).

With these general points in mind, we turn to discussing the four frames in more detail.

The Content Frame

As mentioned earlier, the content frame encompasses knowledge having to do with the particulars of a domain. At the heart of a content frame lies the concepts most central to the particular subject matters within the domain. In the case of elementary Newtonian physics, for example, these would include such notions as force, mass, velocity, and acceleration. In the case of elementary programming, these central concepts would include the notions of variable, expression, assignment statements, loops, and so on. In the case of elementary algebra, concepts like variable, expression, equation, solution, and so on, would be central. Also important to the domain is what might be called "mapping schemes" that associate concepts with referents. Thus, for example, one has to associate mass or acceleration with the right sorts of phenomena in the world in order to handle elementary physics effectively.

The content frame can be faulty in a number of ways. Without pretending to be exhaustive, or to offer formal categories, we describe several kinds of difficulties.

Naive, underdifferentiated, and malprioritized concepts. As has been widely recognized, students do not approach subjects new to them with empty minds. They bring preconceptions that often rival and override those of the topic itself. For some examples from physics, impetus-like conceptions of motion have been found in students of physics in numerous experiments (e.g., Clement, 1982, 1983; McCloskey, 1983; Ranney, 1987). Underdifferentiation between neighboring con-

cepts such as heat and temperature is commonplace, and indeed was a problem in the history of physics (Wiser & Carey, 1983). DiSessa (1983) has pointed out that sometimes the problem can be one not so much of mistaken concepts as malprioritized concepts. For example, to novices, rigidity is as salient a property of matter as springiness, whereas, for a physicist, springiness plays a far-reaching explanatory role, whereas rigidity is a never-achieved limiting case of little interest.

Difficulties in accessing knowledge. Freshly acquired knowledge is likely to be "inert," particularly if the knowledge was obtained in a didactic fashion (Bransford, Franks, Vye, & Sherwood, 1986). To offer an example from programming, experiments with novice programmers reported by Perkins, Martin, and Farady (1986) and Perkins and Martin (1986) revealed that a high percentage of their knowledge was inert: While students commonly seemed to show lack of a relevant knowledge structure, simple nonspecific prompts often led to them recovering the relevant knowledge and proceeding correctly. In other words, they possessed the knowledge but did not initially retrieve it.

Sometimes it is the everyday knowledge rather than the new knowledge that is not accessed. For an example from physics, youngsters who have learned that thermometers measure temperature may lose track of the connection between what the thermometer measures by technical means and their own sense of hot and cold. They will expect thermometer readings to add to yield twice the temperature when two cups of water at the same temperature are combined, never considering their own common-sense knowledge that the "feel" of the water will be the same (Strauss, 1986).

Problems of garbled knowledge. Newly acquired knowledge commonly gets mixed up in various ways as well. To mention a few examples, beginning students of physics may recognize friction as a force operative when one object is moving against another, but not when the object is stationary against the other (Roth & Chaiklin, 1987). Students may attribute both impinging force and reaction force to the same object, rather than the reaction force to the "supporting" object (Anzai & Yokoyama, 1984). In programming, students may import elements from one command into the midst of another (Perkins & Martin, 1986). In arithmetic, as mentioned earlier, students attempt diverse variations on the rules of arithmetic in order to "repair" the situation when they do not quite know what to do (Brown & VanLehn, 1980; VanLehn, 1981).

Considerations like those raised above help us to understand why students display the difficulties that they do. Naive concepts naturally will rival the new ones out of the textbook. A recently acquired knowledge base is likely to contain considerable garbled and inert knowledge. Yet there is more to puzzle about than these considerations in themselves explain. With the problem-solving frame in mind, why do students not struggle harder to retrieve or work around their inert knowledge and to disentangle their garbled knowledge? With the epistemic frame in mind, why do students often cling stubbornly to naive concepts even when they have been shown demonstrations and given arguments that reveal profound difficulties in those naive concepts? With the inquiry frame in mind, why do students not display a more critical and creative spirit in questioning their first construals?

The import of these questions is straightforward. Although misunderstandings of particular domain concepts, rules, and laws necessarily involve the content frame, they appear to be exacerbated by weaknesses in the other frames. These weaknesses

would naturally include higher order versions of the kinds of difficulties just discussed, problems of underdifferentiation, garbled knowledge and so forth. But they also include weaknesses reflecting specifically the focus of each noncontent frame—problem-solving, epistemic, and inquiry. We turn now to examining weaknesses characteristic of the noncontent frames.

The Problem-Solving Frame

Ideally, the problem-solving frame incorporates a wide range of problem-solving knowledge, from that of general attention and progress monitoring to that of domain specific strategic knowledge. The problem-solving frame includes rather general heuristics such as breaking problems down into manageable parts, regulating time spent on any one solution path, and seeking alternative paths. (Brown, Bransford, Ferrara, & Campione, 1983; Polya, 1954; Scardamalia & Bereiter, 1985; Schoenfeld, 1980, 1985). Reflective use of these heuristics, however, depends on considerable content frame knowledge. Also, some problem-solving methods are domain-specific, for example mathematical induction or, in the case of Newtonian physics, the use of free body diagrams. Finally, the problem-solving frame should also include supportive beliefs about and attitudes toward the problem-solving process.

Unfortunately, however, the novice's problem-solving frame may instead be stocked with a number of marginally productive or counterproductive strategies, attitudes, and beliefs. Again without pretense of exhaustiveness or formal categories, here are some cases in point.

Trial and error. Undisciplined trial and error methods are surprisingly common. In mathematics, Schoenfeld (1985) cites the example of college freshmen (having completed a semester of college calculus) who, given a task of geometric construction, guessed incorrectly a solution within a minute of reading the problem, kept guessing until they came up with a reasonable solution on the third conjecture, but were unable to say why the solution worked, offering only, "It just does."

Perseveration and quitting. Another common hazard is perseveration with an approach that is yielding no real progress on the problem. Again, Schoenfeld (1985) describes examples of this behavior. Quitting is the flip side of perseveration. The behavior of sizing up a math problem and quitting at once if no obvious path to a solution presents itself is commonly noted. In programming as well, we have observed both perseveration in an approach and the tendency to quit (Perkins, Hancock, Hobbs, Martin, & Simmons, 1986).

Proceeding on a guess. As discussed in the case of the radical, students who cannot recall just what the rule is, frequently make plausible conjectures and then proceed on that basis, doing nothing to test their conjectures. They thus reason by analogy to generate a possibility—an intelligent move—but fail to deploy any kind of filter to check their possibility.

Stock responses. One common tactic students adopt for coping with the flood of new information as they encounter a new subject, is to develop template-like responses to particular cases, without grasping the underlying principles. Their formulaic thinking shows up when, faced with a somewhat new situation, they respond in a stereotyped way. For example, one student of programming who had recently studied FOR-NEXT loops felt that a problem really ought to include one and quite correctly incorporated a FOR-NEXT loop of the form FOR N = 1 to 1

(Perkins & Martin, 1986). Similarly, students in arithmetic commonly develop stereotyped responses to key terms in word problems—"less" means subtract, "times" means multiply, and so on. For example, Lester (1985) describes a study in which elementary students were asked to figure out how many chickens and how many pigs there were on a farm in which there were 18 animals who had 52 legs in all. Many of the students solved the problem by adding 18 and 52. When questioned about their answers, a typical response was something like, "It asks 'how many in all' so you add."

Equation cranking. In this style of problem solving, students note what needs to be derived, seek equations that yield those results, and work backward toward the givens until they find a chain of equations that will bridge from givens to solution (cf. Chi, Feltovich, & Glaser, 1981; Chi, Glaser, & Rees, 1982; Larkin, McDermott, Simon, & Simon, 1980; White & Horwitz, 1987). In part, this pattern reflects content frame difficulties; novices lack the insight into the domain to assemble qualitative models as a basis for reasoning forward from givens to unknowns, the more typical expert path. At the same time, one would hope that part of the ideal problem-solving frame would be a disposition to seek such paths, using the limited knowledge at one's disposal to understand the problem qualitatively as best one could.

The Epistemic Frame

The epistemic frame focuses on general norms having to do with the grounding of the concepts and constraints in a domain. For example, in physics, one ought to have a theory that is consistent with the evidence; in math, theorems that follow deductively from the axioms involved should accord with one's formal manipulation of those concepts using logic and mathematics. In contrast with many areas of life, the hard sciences, mathematics, and programming demand extraordinarily high standards of coherence between models and what they describe and within models themselves. Unfortunately, most students have not developed the hypersensitivity to coherence required by these technical domains. Some of the weaknesses in the novice's epistemic frame are as follows:

Intuitions mask contrary observation. Expectations based on naive intuitions and prior practice not uncommonly modify what one sees or recollects. For example, diSessa (1983) notes that people typically analyze the question, "How would blocking the intake of a vacuum cleaner affect the sound of the motor," by applying an intuitive concept (in diSessa's terms, a p-prim or phenomenological primitive) in which greater resistance means greater strain, implying a lower-pitched sound. In fact, the pitch goes up, but diSessa's interview subjects frequently reported *remembering* that the pitch went down, in confirmation of their faulty analysis.

Intuitions have priority over internal coherence. The notion that objects of different masses fall at different speeds lacks coherence, as Galileo's famous argument established. The notion that a book on a table pushes on the table, but the table does not push back on the book, also does not yield a coherent analysis: There is no physical basis local to the interface between things that permits determining from which direction the force is coming (cf. Clement, 1987a, 1987b). The notion that one can have an average of 1.5 offspring per family may be rejected as nonsensical even though the mathematical meaning of average makes it perfectly coherent (cf. Strauss, 1986). The notion that a computer program "knows" what

input values should go into variable names like LARGEST dies hard, even though students know in principle that the choice of variable names is theirs, a point incoherent with such wisdom on the part of the computer (cf. Sleeman et al., 1986). Such examples suggest that people commonly fail to notice the incoherencies in their intuitive mental models, and often, when incoherencies are brought to their attention, the incoherencies simply do not appear very important. The robust intuitive model seems worth preserving in the face of a few minor discrepancies.

The grounding of the domain's rules is neglected. Multiplication is distributive over addition in algebra, but the square root is not distributive. Air resistance aside, bodies of different masses fall at the same speed. Where do such rules come from? As noted earlier, novices may easily view such rules as "rules of the game," something someone figured out sometime that one just has to learn. As noted earlier, however, the grounding of rules of algebra essentially lies in rules of arithmetic. Checking a rule of algebra against arithmetic is not merely using a conventional trick, but doing just the right thing—turning to the epistemological foundation of algebra.

Confirmation bias. The tendency to confirm preconceptions emerges strongly and clearly in work on naive physics. One would hope that sophistication in math, programming, and the sciences would bring with it a general caution about preconceptions. And perhaps that happens to a degree. However, it is not rare to see experts exhibiting problems of confirmation bias not unlike those that plague novices, but on a more sophisticated plane. For an example from the history of physics, Wiser and Carey (1983) discuss how scientists exploring an early model of temperature with admirable methodology persistently overlooked puzzles the data posed for their theory. For a contemporary example, most individuals with considerable training in physics conclude that the pressure at the bottom of a milk bottle is constant regardless of whether the cream is distributed or has separated out at the top. In fact, the pressure changes, but a robust elementary physics schema that says roughly that pressure distributes itself in all directions overrides other reasoning (Cohen, 1975; Johnson, Ahlgren, Blount, & Petit, 1980).

In the case of the hard sciences, it is commonplace to view empirical inquiry as the epistemological foundation. This, of course, makes the foundation inaccessible to most students who are in no position to go out and do experiments. We suggest, however, that in fact, a significant part of the foundation of physics, for example, lies in logical coherence and in coherence with gross features of the world, rather than in agreement with the fine structure of the world. For instance, Galileo's argument that objects of any mass fall at the same speed depends basically on logic and on our intuition that snipping a string between two objects is not going to make *that* much of a difference in their rates of fall. The proportionality of F with m in $F = ma$ can be justified in the same way. Inverse square laws can be conceptualized and justified in terms of a flux concept.

To be sure, such justifications await for final verification on empirical evidence, but they can easily precede it and, in many cases at least, the world would be a very strange place if the principles did not hold up. They also frequently have the advantage of providing more compelling understandings of the phenomena concerned than does mere evidence. They show not *that* something happens empirically to be the case but *why* it almost has to be the case. Accordingly, the notion that the epistemological foundations of physics and other hard sciences are empirical

through and through, requiring mountains of data, does mischief by depriving students of an important intellectual resource.

The Inquiry Frame

The inquiry frame is the most ambitious and perhaps hardest to cultivate through education of the four discussed. The inquiry frame encompasses knowledge and attitudes having to do with (a) extending a theory or framework beyond its usual scope and (b) challenging elements of a theory or framework. Such patterns of thought are not so commonly found, even in experts in a field. On the contrary, one often encounters patterns like the following:

No problem finding. Even elementary mathematics, science, and programming provide enough information for students to engage in problem-*finding* activities, where they formulate or participate in formulating the problems to be addressed (re mathematics, see Brown & Walter, 1983; Schwartz & Yerushalmy, 1987). Students, however, show little tendency to engage in problem finding and, indeed, conventional schooling offers few opportunities for such activity. This is unfortunate because evidence suggests that a disposition toward problem finding relates strongly to creative productivity (Getzels & Csikszentmihalyi, 1976; Mansfield & Busse, 1981).

Academic applications only. It is very easy for technical knowledge to remain encapsulated in academic contexts, rather than becoming a window on the world in general. For an extreme example, Richard Feynman (1985) writes eloquently of his experience as a visiting professor in a culture with a strong tradition of rote education. Students would memorize definitions of abstract physical concepts and even master textbook problem solving, but have no idea what ordinary events and objects in the world the abstractions described.

No venturing. In any number of ways, it is possible to go venturing beyond the boundaries of a theory or framework. For example, the same sort of "what if not" question that serves so well to generate problems within a framework (cf. Brown and Walter, 1983) can also challenge the framework itself. Or, for a historical instance of venturing, Einstein was notorious for reaching beyond the boundaries of turn-of-the-century physics not in response to compelling empirical flaws but for aesthetic reasons having to do with elegance and simplicity (cf. Holton, 1973). Such a spirit of inquiry goes beyond shunning confirmation bias because it involves venturing outside a theory even when the theory is functioning quite adequately in some sense. Note that in principle, youngsters could explore variations that contrasted with their own naive theories, just as professional mathematicians and physicists might with state-of-the-art theories. Such venturesomeness, however, is rare in youngsters *and* professionals, understandably so in light of its challenges.

Comparison with Related Frameworks

In the following section, we use our framework to discuss several general patterns of misconception in science, mathematics, and computing. Our framework, however, is not the only recent effort to make some broad statements about levels of knowledge involved in the learning of difficult subject matters. Before applying our framework, let us examine how it relates to some others.

Schoenfeld (1985, in press) discusses four factors that figure in mathematical understanding and problem solving in mathematics and other areas: *resources*

(knowledge base of facts and concepts), *heuristics, control* (metacognitive monitoring and control of problem solving), and *belief systems* (broad beliefs about the nature of mathematics and mathematical inquiry). Collins and Brown (1988), discussing "cognitive apprenticeship," emphasize the importance of learners acquiring four kinds of knowledge: *domain knowledge* (facts and concepts), *heuristic strategies, monitoring strategies,* and *learning strategies.* Finally, Posner, Strike, Hewson, and Gertzog (1982), proposing a model of conceptual change, highlight a number of conditions for change, such as dissatisfaction with a current concept, perceived plausibility of a new concept, and perceived fruitfulness of a new concept. They also emphasize certain features of the learner's "conceptual ecology," for instance, epistemological commitments about the nature of evidence and the importance of parsimony in a theory, and metaphysical beliefs such as faith in the orderliness of nature.

How do these frameworks compare with the framework presented here? Our framework and those of Schoenfeld and Collins et al. acknowledge the importance of knowledge base, problem-solving heuristics, and metacognitive control. In the three frameworks, the first or "bottom level" category concerns knowledge base. Whereas the schemes of Schoenfeld and of Collins and Brown provide separate categories for heuristics and metacognitive control, our scheme allows for heuristic and metacognitive elements within any of our four frames, as described earlier. Beliefs about problem solving, such as "If you can't solve it in five minutes, you can't solve it at all," fall in Schoenfeld's belief systems category but in our problem-solving frame, as described earlier. Other beliefs about a domain fall in our epistemic or inquiry frames.

Why do we include *within* each frame a distinction among strategies, beliefs, and metacognitive control, matters than Schoenfeld and Collins and Brown recognize through separate top-level categories? Because, in our view, the contrast among the three is orthogonal to that among content, problem-solving, epistemic, and inquiry frames. For instance, one can have strategies for, beliefs about, and self-monitoring practices regarding problem solving. Likewise, one can possess epistemic standards, strategies for applying them, and self-monitoring practices for policing their application. Even at the content level, one can speak of beliefs that constitute content, strategies for managing content (hierarchical organization, highlighting, etc.), and monitoring one's content knowledge.

In general, one set of contrasts addresses the form of the knowledge in question—strategic, background beliefs, autoregulative—whereas the four frames address what the knowledge in question concerns—matters of content, problem solving, epistemology, or inquiry. We thus believe that the schemes of Collins and Brown and of Schoenfeld mix dimensions somewhat. We also believe that the epistemic and inquiry frames have great importance and need to be recognized explicitly. At the same time, we acknowledge that certain matters we discuss under the epistemic and inquiry frames have a presence in their schemes, for instance in Schoenfeld's "beliefs" category. We also note that autoregulation may well depend on a single unified mechanism, although one with particular applications in each of the four frames.

Another point of contrast concerns learning. Collins's and Brown's scheme includes a *learning strategy* category, whereas the conceptual change model of Posner, et al. (1982) mostly addresses conditions for accommodating (in contrast

with assimilating) a new concept. We defer comparisons regarding learning until a later section, where we discuss the implications of our framework for educational practice.

Patterns of Misunderstanding

Up to this point, we have discussed the four frames separately in interpreting misunderstandings. Many cases of misunderstanding, however, reflect a distinctive "pattern of misunderstanding," with a characteristic profile of shortfalls across the four frames. In this section, we describe three of the most important to illustrate the general idea: naive, ritual, and Gordian patterns of misunderstanding.

Naive concepts. One of the most typical patterns apparent in many domains might best be called a naive pattern of misunderstanding. This syndrome characterizes the thinking of many novice students. Such students are typically relatively uninformed; the misunderstandings emerge prior to much formal instruction on the topic in question. Consider an example mentioned before, for instance: Students often take the position that, although a book on a table pushes down on the table, the table does not push up on the book. The students perceive no room for a reaction force, because, based on their experience with the real world, the table is "rigid." When the suggestion is raised that the table might be springy after all, students commonly think otherwise. If it is argued on logical grounds that the notion of one-sided forces makes no sense, students may not see the argument or take the view that is too finicky (cf. Brown & Clement, 1987; Clement, 1987a, 1987b; Schultz, Murray, Clement, & Brown, 1987).

With this example at hand, how can one characterize the naive pattern? With respect to the content frame, students simply lack a concept or a priority among concepts that they might obtain with further instruction. Thus, in the example of the book and the table, one sees the novice treating "rigidity" as a concept with priority over "springiness," (cf. diSessa, 1983). In contrast, the expert recognizes springiness as a much more powerful explanatory tool.

In part, then, the novice's problem is simply one of editing initial conceptions in light of new knowledge as it comes along. This in itself, however, does not explain why naive notions often are so robust, another characteristic of the naive pattern. Of course, cognitive load and related developmental factors are likely to be responsible in part (cf. Brainerd, 1983; Case, 1984, 1985). It's also helpful, however, to consider the role of the epistemic and inquiry frames in a naive concept.

Regarding the epistemic frame, counterarguments are disregarded in part because the students have not yet recognized that in science the rules of the game demand that things hang together in an extraordinarily coherent fashion. Small anomalies are simply unacceptable. Moreover, if one can account for a wide range of phenomena with springiness, and treat rigidity as a kind of limiting case, this parsimony is good scientific coin; the intuitive reality of rigidity is no longer compelling.

As to the inquiry frame, ideally it would be easy to provoke the naive theorist to adopt a venturesome attitude and explore a variety of formulations. However, many youngsters—perhaps most—evince more a spirit of conviction than a spirit of exploration in their theorizing.

All this is part of what might be called the "culture of science"; a culture that cannot be taken for granted and that students typically have had little chance to

assimilate. Posner and his colleagues have discussed the difficulties inherent in achieving student conceptual change when the epistemological commitments of the student differ from those of the scientific community (Posner, Strike, Hewson, & Gertzog, 1982). They note that most fields maintain explanatory ideals; specific views concerning what counts as a good explanation in a particular field. In addition, they describe what appears to be a consensus across scientific fields for what forms the character of successful knowledge, for example, elegance, economy, parsimony, and not being *ad hoc.*

We have not yet mentioned the problem-solving frame. This frame is not a central element in the naive pattern, simply because the student is not yet at the level of seriously engaging in technical textbook problem solving. As will be emphasized in the next pattern, naive intuitions, however, can continue to affect performance even in students with considerable technical problem-solving skills. In sum, a naive concept points to a shortfall across all frames of knowledge. Poor performance is the consequence of misunderstandings in the content frame protected from revision by epistemic and inquiry frames that lack "the culture of science."

Ritual concepts. In contrast to a naive pattern of misunderstanding, what we call the "ritual pattern" arises in students who have undergone considerable formal instruction and may well have developed a high degree of technical problem-solving skill in dealing with textbook problems. At first glance, the student seems to have quite a respectable understanding. Yet further analysis establishes that in fact the student applies knowledge in a somewhat ritualistic fashion, and proves unable to deal with novel situations even when the knowledge base should be more than adequate to the task.

The ritual knowledge syndrome has three notable features. First, the student is typically adept at equation cranking as a means of solving technical problems, exhibiting a clear grasp of many of the intricacies of the notational systems in question, however, the student displays little sensitivity to the "deep structure" of problems in the domain (cf. Chi, Feltovich, & Glaser, 1981; Chi, Glaser, & Rees, 1982; Larkin, McDermott, Simon, & Simon, 1980; Schoenfeld & Herrmann, 1982). Second, when tasks are posed that do not suit the equation-cranking approach, unrevised and incorrect intuitive knowledge commonly overrides the student's technical knowledge. For example, many students who have received significant physics instruction, even at the college level, display misconceptions when qualitative problems are posed (Clement, 1982, 1983; McCloskey, 1983).

For instance, students marking the forces at work as a tossed ball rises, peaks, and falls commonly identify a nonexistent "impetus" force that sustains the upward rise, matches gravity at the peak, and disappears or at least becomes less than gravity on the fall. This is an interesting misconception in that many of the students displaying it have studied Newton's laws of motion and ostensibly could apply the laws to reason out the problem. Moreover, students could in fact take the given problem and cast it algebraically, finding no force at play other than gravity in analogy to other problems they have done, for instance, finding how high the ball would rise given a certain upward momentum from the toss. Yet students resort, instead, to a reliance on naive intuitions and ignore the scientific knowledge at their disposal.

A third feature of the ritual pattern of misunderstanding addresses the flip side

of the situation described above: Instead of unsound intuitions overriding technical knowledge, overgeneralized technical knowledge dominates a situation. Consider the milk bottle example mentioned earlier, in which even professional physicists argue that the pressure at the bottom of a milk bottle is no different with the cream dispersed than with the cream separated out at the top. Here, a sophisticated schema about pressure proves overgeneralized and prompts an incorrect response (Cohen, 1975; Johnson, Ahlgren, Blount, & Petit, 1980).

How do the four frames of knowledge inform us in the case of the ritual pattern? In the content frame, one finds a much more sophisticated verbal knowledge base than in the naive pattern. However, the intuitive imagistic level of students' content understanding has hardly been touched. Naive conceptions persist underneath and resurface when the student does not immediately see a quantitative solution to a problem. Also, students may have acquired overgeneralized technical schemas that generate errors. In the problem-solving domain, contrary to the naive pattern of behavior, students may in fact exhibit quite sophisticated performance in technical problem solving. However, they tend to base approaches to problems through equation cranking based on surface features rather than on the "deep structure" of the problems—the basic principles to relevant to their resolution. Knowledge in the epistemic and inquiry frames may be hardly more developed than in the naive pattern; students do not display much sense of the epistemic roots of principles nor take a critical stance toward their intuitions.

Gordian concepts. Finally, let us examine what we will call a "Gordian pattern" of misunderstanding, so named for the proverbial Gordian knot. The Gordian pattern occurs when experts elaborate a theory with serious undetected errors. In this case the four frames seem to be quite well developed, and there is an expectation that the resultant theories are well grounded. Yet, for all that, grossly erroneous conclusions are drawn from data. Consider, for example, the work of 17th century experimenters in the area of thermal phenomena (Wise & Carey, 1983). This group of scientists had developed a theory of thermodynamics based on a mechanical model. In their Source-Recipient model, there was no differentiation between heat and temperature; rather, heat and cold were conceived of having intrinsic force or strength, and were seen as two separate concepts. This led them to concentrate empirical research on seeking out the mechanical effects of heat and cold, ignoring the possibility of an intervening variable (temperature) to link heat to volume expansion. This led them again and again to miss anomalies in their data or to reinterpret them to suit their theory.

In this Gordian pattern, the four frames played out in an interesting way. The content frame was constructed from a set of principles and notational system accepted by the scientific community at large. The problem-solving techniques were advanced. In general, the epistemic frame was well developed also: The scientists certainly took care to justify their claims with observational data. However, confirmation bias appeared in the inquiry frame, even to the point of misreading the significance of data.

To be sure, exactly what one's posture as a scientist ought to be toward anomalies in data with reference to current theory is a dilemma: Some argue that new theories need to be protected for a while from the rigorous test of conformity with data so that they have time to grow (e.g., Feyerabend, 1975). However, at least it seems desirable to know that the anomalies are there, even if one defers considering them.

Learning and Learning Better

So far, we have offered an integrated view of students' misunderstandings in science, mathematics, and computing. Now we engage the question of learning: How do learners acquire these misunderstandings, and what means of instruction might serve students better?

It is natural to wonder whether the model articulated here, with its four frames and the naive, ritual, and Gordian patterns of misunderstanding, requires some special account of learning. We suggest that it does not. As emphasized earlier, the content of the four frames consists in declarative and procedural knowledge at various levels of abstraction, including beliefs, algorithms, strategies, and self-monitoring practices. Elements of, say, the epistemic frame are acquired through much the same processes that serve learning in the content frame—a mix of direct instruction, personal discovery, proceduralization, and so on. With this eclectic stance on learning in mind, what can be said about the sources of and remedies for students' misunderstandings, from the point of view of the present framework?

The Disposition Toward Misunderstandings

Before considering the role of formal education, it is important to note that several factors dispose learners toward the naive, ritual, and Gordian patterns of misunderstanding. First of all, as mentioned earlier, extensive evidence has accumulated showing that students evolve for themselves "naive" notions about a number of scientific and mathematical topics, such as dynamics (cf. Clement, 1982; McCloskey, 1983). These naive theories, at least in part, appear to reflect straightforward and often pragmatically useful generalizations about the way the everyday world behaves. For instance, the straight-line motion at a constant velocity of Newton's first law rarely is seen in ordinary circumstances.

Such naive theories also reflect the failure to make certain discriminations that are genuinely subtle, such as the distinction between heat and temperature (Wiser & Carey, 1983), or the point that zero velocity does not imply zero acceleration. In addition, they appear to reflect "phenomenological primitives" such as rigidity that, because of salience in everyday life, may mask the explanatory breadth of other concepts such as springiness (diSessa, 1983). In general, it is important to remember how much investigative effort went into our sophisticated understandings in science, mathematics, and computing. There is every reason to expect youngsters, when they think about such matters at all, to fall into naive understandings. Given a tendency toward many naive conceptions, the persistence of such conceptions is explicable in at least two ways. First of all, there is the tendency to assimilate rather than accommodate. For example, youngsters taught that the world is round tend to construct planar versions of this roundness, such as a flat disk, thus reconciling the notion that the world is round with their perceptual experience (Nussbaum, 1985). Also, the process of knowledge compilation has a conservative tendency, as productions are developed that reflect the specific context of learning (Anderson, 1983). Thus, instruction and experiences that might challenge a naive theory easily become compartmentalized, and the challenge never occurs. Instead, a ritual pattern of misunderstanding evolves.

Finally, developmental factors relating to the epistemic and inquiry frames need to be taken into account. Recall that the epistemic frame concerns sophistication

about canons of verification in domains. Accordingly, it includes such classic Piagetian considerations as control of variables and hypothetico-deductive reasoning. It also encompasses postures toward knowledge ranging from dualistic to constructivist, as outlined in Perry's scheme (1970), for example, or in the developmental stage theory of Kitchener (1986) or Basseches (1984). Regarding the inquiry frame, Arlin (1986) has proposed a developmental progression centered on the role of "problem finding" in inquiry.

Research organized around any of these schemes shows that sophistication in matters of epistemology and inquiry is a relatively late developmental achievement. We do not argue the strong Piagetian position that certain cross-domain logical structures are necessary conditions for the attainment of domain-specific understandings. But we do suggest that such higher order understandings are important facilitators of conceptual change, as argued earlier and also as suggested by the conceptual change theory of Posner et al., 1982.

How Conventional Instruction Falls Short

Whatever the natural tendency toward the naive, ritual, and Gordian patterns of misunderstanding, it has always been the job of education to work against such trends. Why does conventional educational practice seem so remarkably unsuccessful?

Here we resort to what might be called a "first order theory of instruction." This theory—almost too simple to merit the name—nonetheless offers ready explanations for many of the ills of conventional instructional practice. The first order theory simply says: *People learn much of what they have a direct opportunity and some motivation to learn, and little else.* By "direct opportunity" we mean opportunity in interaction with teachers, materials, or peers to hear about, see demonstrations of, and engage in guided and independent applications of the knowledge in question. The style of instruction could be highly constructivist, direct instruction, cognitive apprenticing, or something else. But instructional style is subordinate to opportunity and motivation: Unless the instruction provides opportunity and motivation, little learning is likely; if the instruction does, considerable learning is likely.

Straightforward, though, this first order theory is, educational practice widely ignores it. We have argued that learning with understanding depends on all four frames and their interrelations. However, students normally do not have direct opportunities to learn about these frames and their interrelationships, nor are they, perforce, rewarded for doing so. In the typical school setting, the inquiry frame gets virtually no attention at all. On the contrary, subject matters are taught as received knowledge not subject to challenge. Moreover, the curriculum is dominated by stereotypical "school problems"—school algebra, school physics, school programming, and so on—with students little encouraged to map the content into applications beyond school problems. The epistemic frame fares only a little better. To be sure, in some kinds of mathematics instruction—Euclidean geometry for example—attention is paid to the question of proof. Also, in some science instruction, key experiments are celebrated and, when accessible, reproduced in the school laboratory. However, all this typically has the character of a ritual exercise where rote learning dominates.

The problem-solving and content frames are the focus of most classroom

instruction. On the positive side, students certainly receive exposure to plenty of content and get extensive practice in solving problems. Better students may become quite good at solving textbook problems. On the negative side, however, conventional education offers little direct instruction in heuristics and problem-solving management, adopting an almost pure demonstration and practice approach without attention to the metacognitive side of problem solving. As to the content frame, the strategic side of even fact learning, for example, heuristics for memorizing, receives little attention, and it is widely recognized that most curricula attempt to cover far too much content at the cost of depth of understanding.

In summary, conventional education gives most attention to the content frame, next most to the problem-solving frame, next to the epistemic frame, and finally and hardly at all to the inquiry frame. But even the content frames does not fare all that well. Moreover, the thrust of our argument has been that teaching for understanding requires attention to all the frames; one cannot just teach content and expect understanding.

Is Instruction Encompassing Several Frames Possible?

We have argued that, for learning with understanding, students need instruction encompassing several frames. But is such an integrative kind of instruction technically possible and practically manageable? An affirmative answer to both these questions come from considering several examples from the literature.

For instance, Alan Schoenfeld has conducted various experiments in the direct teaching of heuristics and problem-management strategies for mathematical problem solving (Schoenfeld, 1980, 1982, 1985; Schoenfeld & Herrmann, 1982). In experiments with a college-level intensive course, Schoenfeld has demonstrated striking improvements in students' mathematical problem-solving abilities, with transfer to problems of unfamiliar types. The results also showed alleviation of the ritual pattern of misunderstanding, part of which, it will be recalled, involves missing the underlying principles in problems: The investigators demonstrated changes in students' classification of problems in the direction of expert mathematical problem solvers (Schoenfeld, 1982; Schoenfeld & Herrmann, 1982).

These efforts plainly address the problem-solving frame in relation to the content frame. For a related example from programming, we and colleagues have constructed a "metacourse" for elementary programming instruction. This metacourse consists in supplementary lessons that, interleaved with a teacher's normal instruction, provide mental models of the computer and problem-solving strategies tuned to the needs of programming. Extensive field testing has shown that this "metacourse" has a strong impact on students' programming performance (Perkins, Schwartz, & Simmons, in press).

We mentioned earlier a misconception concerning Newton's third law: Many students of high school physics initially tend to hold that, although a book on a table pushes down on the table, the table does not push up on the book—there is no reaction force because the table is rigid and cannot push back. Addressing this naive pattern of misunderstanding, John Clement and his colleagues have explored a Socratic classroom procedure in which the teacher moderates a discussion exploring the logic of this position: What about a book sitting on a spring? What about a bendy table? At the opposite extreme, what about a fly standing on a road? Through this interactive conversation, many students come to see that coherence

is better served by the position that everything bends a little and there is always a reaction force; otherwise, arbitrary boundaries must be drawn (Brown & Clement, 1987; Clement, 1987a, 1987b; Schultz, Murray, Clement, & Brown, 1987). This effort models patterns of reasoning about validity—the epistemic frame—as a way of helping students see the logic of the Third Law.

In a somewhat related manner, White and Horwitz (1987) have constructed and demonstrated the efficacy of a microworld designed to give students a better grasp of Newtonian motion. In dealing with this microworld, students have to consider generalizations that may challenge their own initial conceptions, use these generalizations to make predictions, and test them, thus engaging the inquiry and epistemic frames.

Another example laying special emphasis on the inquiry and epistemic frames concerns the *Geometric Supposer,* a piece of software developed by Schwartz and Yerushalmy (1987), designed to restore an inquiry process to instruction in Euclidean geometry. The *Geometric Supposer* makes geometric constructions easy by providing computer assistance in dropping altitudes and angle bisectors, adding parallels, and so on. It also permits automatically "replaying" a construction with different starting points—a new triangle for instance—to allow examining multiple cases for similarities. In geometry classrooms, the *Geometric Supposer* provides a tool with which students explore possible geometric relations, devise conjectures, test their conjectures informally with the *Geometric Supposer,* and then often attempt formal proofs. Students routinely rediscover standard theorems rather than learning them out of the text, and, from time to time, students have formulated unknown theorems.

Toward a Pedagogy of Understanding

The cases reviewed show considerable promise, dealing as they do with the content and one or more other frames in the service of deepening students' understanding. Still, one can ask how far such examples go toward a pedagogy of understanding. Of course, we cannot say what an ideal pedagogy of understanding would be like. However, we would argue for the following minimal conditions:

1. Instruction should include all four frames.
2. Instruction should treat the frames not in isolation, but rather in terms of the way each relates to and informs the others.
3. Instruction should involve explicit articulation by teachers and/or students of the substance of the frames and their interrelationships.

The rationale for these conditions is straightforward. Our discussion of the patterns of misunderstanding argued for the importance of the frames and their mutually reinforcing interrelationships. Therefore, the "first order theory of instruction" mentioned above prescribes direct opportunity to learn about the frames and their interrelationships, an opportunity fostered by explicit treatment.

How well do the examples just given measure up against these standards? As to the examples, all dealt with at least two frames in interaction—unsurprisingly, since the examples were selected to illustrate that very possibility. *Explicit* treatment of the noncontent frames, however, is not universally found. For example, the experiments involving the book on the table, for the most part, did not focus directly on the epistemic frame, simply using students' intuitive allegiance to coherence as an epistemic principle to lead students to a Newtonian analysis of the

forces. In contrast, Schoenfeld's instructional method focuses quite explicitly on heuristics of problem solving and methods of managing the problem-solving process.

Earlier, we related the four frames to the cognitive apprenticeship model of Collins and Brown (1988). Now we can assess the extent to which the cognitive apprenticeship model reflects the learning principles given above. Although not laid out as such in the model, the three principles are certainly entailed by the cognitive apprenticeship model. In particular, the model advocates several general methods of teaching heuristic, monitoring, and learning strategies. Most of the methods involve explicit articulation of these strategies in ways that bring them into direct contact with content-level problems. For instance, in *modeling*, a teacher simultaneously demonstrates a thinking process and comments upon its principles. In *reflection*, students are led to look back upon their handling of a task, seek generalizations, and compare their process with expert methods.

While the methods prescribed by the cognitive apprenticeship model certainly accord with our criteria, we recall a point made earlier: The cognitive apprenticeship model does not explicitly recognize the epistemic and inquiry frames and does not distinguish questions of focus of knowledge (content, problem-solving, epistemic, inquiry) from questions of form of knowledge (beliefs, strategies, autoregulation). As noted earlier, the latter forms of knowledge occur within all the frames. In our view, the already very helpful model of Collins and Brown should be clarified and broadened accordingly.

Another contemporary view of learning mentioned earlier was the conceptual change model of Posner et al., 1982. How do the criteria apply to this model? The conceptual change model highlights knowledge in the epistemic and inquiry frames as important to conceptual change in content concepts. For example, epistemic values concerning coherence and inquiry-oriented values concerning the fruitfulness of a revised concept count as important conditions for abandoning an old conceptualization in favor of a new one. As this point makes plain, conceptual change theory certainly holds that the epistemic and inquiry frames inform content-level conceptual change.

In a couple of respects, however, conceptual change theory may not entirely suit the criteria sketched above. Conceptual change theory does not appear to include a position on the importance of explicit instruction involving the epistemic and inquiry frames. Moreover, the interaction with the content frame as envisioned in conceptual change theory seems unidirectional: The epistemic and inquiry frames impact on conceptual advance in the content frame. Our own view is more interactional: Learning in the content and other frames proceeds in parallel, each informing the others in a process of mutual bootstrapping. Also, conceptual change theory is particularly directed to conceptual change involving strongly held misconceptions. In our view, often misconceptions are not so strongly held, so epistemic and inquiry knowledge rather weaker than that laid out by Posner et al., may often suffice for conceptual change.

Conclusion

We have sought an integrated account of students' misunderstandings in science, mathematics, and computing. We have argued that understanding a concept in the content frame is only superficially a matter of content-frame knowledge alone.

Real understanding consists in a web of relationships that connect with content knowledge but also with knowledge in the problem-solving, epistemic, and/or inquiry frames. Failure to recognize this web of interrelationships leads to instruction that allows and even exacerbates the naive and ritual patterns of misunderstanding (the Gordian pattern being more characteristic of entrenched theories in professional practice). In contrast, education that attends to multiple frames and their reciprocal mutually reinforcing relationships should—and in some documented cases does—lead to deeper understanding.

The present model joins with other recent inquiries—for instance the conceptual change theory of Posner et al., 1982, and the cognitive apprenticeship model of Collins and Brown (1988)—in building toward what might be called a pedagogy of understanding, a theory of instruction that describes and prescribes the conditions under which we can expect students to learn difficult concepts with genuine understanding.

Models in this direction seem necessary to revitalize educational practice, especially because educational practice, like the fields examined here, also is subject to reductive misconceptions. Broadly speaking, education tends to be dominated by default assumptions about what knowledge and understanding are and how they are acquired. Just as the default position for many science students is that force varies with velocity, not acceleration, so the default assumption for many involved in the educational enterprise—students, teachers, and curriculum writers alike—is that understanding varies with information and practice.

Of course, *sometimes* these default assumptions are correct. At terminal velocities in a resistive medium, force varies entirely with velocity. Where there are no particular conceptual barriers, understanding is pretty much a manner of information and practice. Unfortunately, both theories miss the essence. Consequently, educating for understanding means not only helping students to remake their concepts of force, fractions, or FOR-NEXT loops but helping the educational community at large to remake reductive concepts of learning and understanding by means of more encompassing, compelling, and accessible theories of instruction.

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Authors

DAVID PERKINS, Co-Director, Project Zero and Lecturer, Harvard Graduate School of Education, Longfellow 315, Appian Way, Cambridge, MA 02138. *Specializations:* the teaching of thinking, informal reasoning.

REBECCA SIMMONS, Research Assistant, Harvard Graduate School of Education, 315 Longfellow Hall, Appian Way, Cambridge, MA 02138. *Specializations:* learning theory, educational uses of technology.